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MATH 323 - Notes 11/5/2021

One last example in Spherical Coordinates

Ex: Compute the volume of a disk of radius $\alpha > 0$

NB: We already did this in Cartesian coordinates but it was nasty!!

Solution: In spherical coordinates $D_\alpha = \{(\rho, \theta, \phi) : 0 \le \rho \le \alpha,\ 0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi\}$

$$Vol(D_\alpha) = \iiint_{D_\alpha \, cart} 1 \, dV_{cart} \qquad dV_{cart} = \rho^2 \sin(\phi)\, dV_{sph}$$

$$= \iiint_{D_\alpha \, sph} 1 \cdot \rho^2 \sin(\phi)\, dV_{sph} = \int_{\rho=0}^{\alpha} \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\phi=\pi} \rho^2 \sin(\phi)\, d\phi\, d\theta\, d\rho$$

$$= \int_{\rho=0}^{\alpha} \int_{\theta=0}^{2\pi} -\rho^2 [\cos(\phi)]_{\phi=0}^{\pi}\, d\theta\, d\rho = \int_{\rho=0}^{\alpha} \int_{\theta=0}^{2\pi} -\rho^2 (-1-1)\, d\theta\, d\rho$$

$$= 2\int_{\rho=0}^{\alpha} \rho^2 [\theta]_0^{2\pi}\, d\rho = 2\int_{\rho=0}^{\alpha} (2\pi)-0\, \rho^2\, d\rho = 4\pi \left(\tfrac{1}{3}\rho^3\right)_{\rho=0}^{\alpha}$$

$$= \frac{4}{3}\pi\alpha^3$$

§16.2 : Vector Feilds

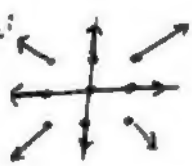
Goal: Study $\vec{v} : \mathbb{R}^n \to \mathbb{R}^n$

Defn: A vector feild on $\mathbb{R}^n$ is a function $\vec{v} : \mathbb{R}^n \to \mathbb{R}^n$

(vector feild may be abbreviated as V.F.)

Ex: $\vec{v}(x,y) = \langle x, y \rangle$ is a v.f. on $\mathbb{R}^2$

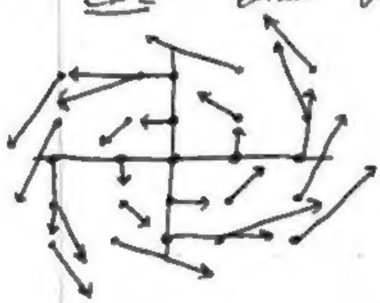
Picture:



$v(1,0) = \langle 1, 0 \rangle \longrightarrow$

NB: in pictures, we shift $\vec{v}(x,y)$ to have tail (x,y)

Ex 2: draw $\vec{v}(x,y) = \langle -y, x \rangle$



$v(0,0) = \langle 0,0 \rangle$ $\quad v(1,-2) = \langle 2,1 \rangle$
$v(1,1) = \langle -1,1 \rangle$
$v(1,0) = \langle 0,1 \rangle$
$v(1,2) = \langle -2,1 \rangle$

Ex: Given any Function $F: \mathbb{R}^n \to \mathbb{R}$, we obtain a vector feild by taking the gradient:

e.g. $F(x,y) = xy$

$\nabla F(x,y) = \langle y, x \rangle$ is the gradient vector feild of F.

e.g. $f(x,y,z) = e^{x+y^2}\cos(x+z)$

$\nabla F(x,y,z) = \langle e^{x+y^2}\cos(x+z) - e^{x+y^2}\sin(x+z), 2ye^{x+y^2}\cos(x+z), -e^{x+y^2}\sin(x+z) \rangle$

is a vector feild on $\mathbb{R}^3$

e.g. $F(x,y) = x^2 + 3xy - xy^2$

$\nabla F(x,y) = \langle 2x + 3y - y^2, 3x - 2xy \rangle$

Terminology: ① A vector feild is conservative when it is the gradient vector feild of some function.

② When $\vec{v} = \nabla f$ is conservative we say f is a potential function for $\vec{v}$.

Obvious Question: Which v.f.s are conservative? are all v.f.s conservative

If $\vec{v}(x,y)$ is conservative, then $\vec{v} = \nabla f(x,y)$

i.e. $\vec{v}(x,y) = \langle f_x(x,y), f_y(x,y) \rangle$

By Clairaut's Theorem, $f_{xy} = f_{yx}$, so for $\vec{v} = \langle v_x, v_y \rangle$ we have

$\frac{\partial}{\partial y}[v_x] = \frac{\partial}{\partial x}[v_y]$ for all conservative v.f.s

Ex: $\vec{v} = \langle -y, x\rangle$ is not conservative! Because...

$$\frac{\partial}{\partial x}[V_y] = \frac{\partial}{\partial x}[\cancel{V_x}] = 1 \neq -1 = \frac{\partial}{\partial y}[-y] = \frac{\partial}{\partial y}[V_x]$$

It turns out this is an "iff" type condition!

Proposition: A vector feild $\vec{v}(x_1, x_2, \cdots, x_n) = \langle V_1, V_2, \cdots, V_n\rangle$ is conservative if and only if for all i, j we have $\frac{\partial}{\partial x_i}[V_j] = \frac{\partial}{\partial x_j}[V_i]$ (i.e.) a v.f. is conservative iff it satasfies Clairaut's Theorem)

NB. A proof of this result follows from the methods I give below

Ex: $\vec{v} = \langle \overset{V_x}{x}, \overset{V_y}{y}\rangle$ Conservative? If yes, potential

Sol: $\frac{\partial}{\partial x}[V_y] = \frac{\partial}{\partial x}[y] = 0$

$\frac{\partial}{\partial y}[V_x] = \frac{\partial}{\partial y}[x] = 0$

To compute the potential: If $\vec{v} = \nabla f$, then $f_x(x,y) = x$ and $f_y(x,y) = y$

$\therefore f(x,y) = \int \frac{\partial f}{\partial x}\,dx = \int x\,dx = \frac{1}{2}x^2 + C(y)$

$\therefore y = f_y(x,y) = \frac{\partial}{\partial y}[\frac{1}{2}x^2 + C(y)] = \frac{\partial C}{\partial y}$

← honest constant

Hence $C(y) = \int \frac{dC}{dy}\,dy = \int y\,dy = \frac{1}{2}y^2 + D$

$\therefore f(x,y) = \frac{1}{2}x^2 + C(y) = \frac{1}{2}x^2 + \frac{1}{2}y^2 + D$ is a potential for $\vec{v}$ for every constant D $\boxtimes$

Ex: $\vec{v} = \langle 2xy, x^2 - 3y^2\rangle$ Conservative? If so, compute potential

Sol: $\frac{\partial}{\partial x}[V_y] = \frac{\partial}{\partial x}[x^2 - 3y^2] = 2x$

$\frac{\partial}{\partial y}[V_x] = \frac{\partial}{\partial y}[2xy] = 2x$

$\therefore \vec{v}$ is conservative, i.e. $\vec{v} = \nabla f$ for some $f(x,y)$. $\frac{\partial f}{\partial x} = 2xy$ and $\frac{\partial f}{\partial y} = x^2 - 3y^2$

$\therefore f(x,y) = \int \frac{\partial f}{\partial x}\,dx = \int 2xy\,dx = x^2y + C(y)$

Hence $x^2 - 3y^2 = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}[x^2y + C(y)] = x^2 + \frac{\partial C}{\partial y}$

← (honest constant)

$\therefore \frac{dC}{dy} = -3y^2$ so $C(y) = \int \frac{dC}{dy} = \int -3y^2 = -y^3 + D$

$\therefore f(x,y) = x^2y + C(y) = x^2y - y^3 + D$ is a potential function for $\vec{v}$ for every constant D $\boxtimes$

Ex $\vec{v}(x,y) = \langle \frac{1}{x+y}, \ln(x+y) \rangle$ Conservative? Potential?

Sol: $\frac{d}{dx}[\ln(x+y)] = \frac{1}{x+y}$ and $\frac{d}{dy}[(x+y)^{-1}] = -(x+y)^{-2}$

$\therefore \frac{\partial v_2}{\partial x} \neq \frac{\partial v_1}{\partial y}$ so v is not conservative $\boxtimes$